

Orthogonal polynomials on a bi-lattice ^{*}

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Abstract

We investigate generalizations of the Charlier and the Meixner polynomials on the lattice \mathbb{N} and on the shifted lattice $\mathbb{N} + 1 - \beta$. We combine both lattices to obtain the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$ and show that the orthogonal polynomials on this bi-lattice have recurrence coefficients which satisfy a non-linear system of recurrence equations, which we can identify as a limiting case of an (asymmetric) discrete Painlevé equation.

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1 Introduction

The classical orthogonal polynomials (Jacobi, Laguerre and Hermite polynomials) are orthogonal polynomials on (an interval) of the real line with a weight function that satisfies a first order differential equation (the so-called Pearson equation) of the form

$$[\sigma(x)w(x)]' = \tau(x)w(x),$$

where σ is a polynomial of degree at most 2 and τ is a polynomial of degree 1. This Pearson equation allows to find many useful properties of these polynomials. There are other families of orthogonal polynomials in the Askey table [5] which live on a linear lattice rather than on an interval. These are the Charlier polynomials ([5, §9.14 on p. 247], [3, Chapter VI, §1])

$$\sum_{k=0}^{\infty} C_n(k; a) C_m(k; a) \frac{a^k}{k!} = a^{-n} e^a n! \delta_{n,m}, \quad a > 0,$$

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which are orthogonal on the lattice \mathbb{N} with respect to the Poisson distribution, and the Meixner polynomials ([5, §9.10 on p. 234], [3, Chapter VI, §3] who calls them Meixner polynomials of the first kind)

$$\sum_{k=0}^{\infty} M_n(k; \beta, c) M_m(k; \beta, c) \frac{(\beta)_k c^k}{k!} = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{m,n}, \quad \beta > 0, \quad 0 < c < 1,$$

which are orthogonal on \mathbb{N} with respect to the negative binomial (or Pascal) distribution. The special case $\beta = -N$ and $c = p/(p-1)$, with $N \in \mathbb{N}$ and $0 < p < 1$ are the Krawtchouk polynomials which are orthogonal on the integers $\{0, 1, 2, \dots, N\}$ with respect to the binomial distribution. The Hahn polynomials are another family of polynomials which are orthogonal on the integers $\{0, 1, 2, \dots, N\}$. Instead of the differential operator it is better for these lattice polynomials to work with the difference operators

$$\nabla f(x) = f(x) - f(x-1), \quad (\text{backward difference})$$

and

$$\Delta f(x) = f(x+1) - f(x), \quad (\text{forward difference}).$$

The weight $w_k = a^k/k!$ for Charlier polynomials now satisfies the Pearson equation

$$\nabla w(x) = \left(1 - \frac{x}{a}\right) w(x)$$

if we define the weight function $w(x) = a^x/\Gamma(x+1)$, so that $w_k = w(k)$. This is of the form $\nabla[\sigma(x)w(x)] = \tau(x)w(x)$ with $\sigma = 1$ and τ a polynomial of degree 1. The weight for Meixner polynomials $w_k = (\beta)_k c^k/k!$ can be written as $w_k = w(k)$ using the weight function $w(x) = \Gamma(\beta+x)c^x/(\Gamma(\beta)\Gamma(x+1))$ and this satisfies the Pearson equation

$$\nabla[(\beta+x)w(x)] = \left(\beta+x - \frac{x}{c}\right) w(x),$$

so that $\sigma(x) = \beta+x$ and τ is a polynomial of degree 1. This Pearson equation (with a difference operator) allows to find many properties of these orthogonal polynomials on the lattice \mathbb{N} . In particular one can find the three term recurrence relation for these polynomials explicitly:

$$-xC_n(x; a) = aC_{n+1}(x; a) - (n+a)C_n(x; a) + nC_{n-1}(x; a)$$

and

$$(c-1)xM_n(x; \beta, c) = c(n+\beta)M_{n+1}(x; \beta, c) - [n+(n+\beta)c]M_n(x; \beta, c) + nM_{n-1}(x; \beta, c).$$

Often it is more convenient to work with the monic polynomials $P_n(x) = x^n + \dots$, for which the recurrence relation is

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x),$$

or with the orthonormal polynomials $p_n(x) = \gamma_n x^n + \dots$, with

$$\frac{1}{\gamma_n^2} = \sum_{k=0}^{\infty} P_n^2(k) w_k,$$

for which the recurrence relation is

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x).$$

The recurrence coefficients for Charlier polynomials then can be seen to be $a_n^2 = na$ and $b_n = n + a$ for $n \in \mathbb{N}$, and for Meixner polynomials one has

$$a_n^2 = \frac{n(n + \beta - 1)c}{(1 - c)^2}, \quad b_n = \frac{n + (n + \beta)c}{1 - c}, \quad n \in \mathbb{N}.$$

In this paper we will investigate generalizations of the Charlier and the Meixner polynomials. In Section 2.1 we will use the weight function $w(x) = \Gamma(\beta)a^x/(\Gamma(\beta + x)\Gamma(x + 1))$ which gives the weights $w_k = w(k) = a^k/((\beta)_k k!)$ on the lattice \mathbb{N} . In Theorem 2.1 we will find a system of non-linear recurrence equations for the recurrence coefficients a_n^2 and b_n and we can identify this system as a limiting case of a discrete Painlevé IV equation. In Section 2.2 we will use this weight on the shifted lattice $\mathbb{N} + 1 - \beta$ and in Theorem 2.2 we will show that the recurrence coefficients satisfy the same non-linear system of recurrence equations, but with a different initial condition for b_0 . In Section 2.3 we combine both lattices to obtain the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$. The orthogonality measure is now a linear combination of the measures on \mathbb{N} and $\mathbb{N} + 1 - \beta$ and surprisingly, Theorem 2.3 shows that the recurrence coefficients again satisfy the same non-linear system of recurrence equations, but with a different initial condition for b_0 . In all these cases, the initial conditions for b_0 are given by ratios of modified Bessel functions. In Section 3 we deal with a similar generalization of the Meixner weight and we investigate the weight function $w(x) = \Gamma(\beta)\Gamma(\gamma + x)a^x/(\Gamma(\gamma)\Gamma(\beta + x)\Gamma(x + 1))$. In Section 3.1 we use this weight on the lattice \mathbb{N} where it gives the weights $w_k = w(k) = (\gamma)_k a^k/((\beta)_k k!)$. In Theorem 3.1 we will show that the recurrence coefficients of the corresponding orthogonal polynomials satisfy another non-linear system of recurrence equations, which we can identify as a limiting case of an asymmetric discrete Painlevé IV equation. In Section 3.2 we use the same weight function but now on the shifted lattice $\mathbb{N} + 1 - \beta$. The recurrence coefficients of the corresponding orthogonal polynomials satisfy the same non-linear system of recurrence equations, but with a different initial value for b_0 . Finally, in Section 3.3 we study the orthogonal polynomials on the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$ with respect to a linear combination of the measures on \mathbb{N} and $\mathbb{N} + 1 - \beta$, and again the recurrence coefficients satisfy the same system of non-linear recurrence equations, but with a different initial value for b_0 . The initial conditions in Section 3 are all given by ratios of confluent hypergeometric functions.

We hope this paper is of interest to people studying orthogonal polynomials because we introduce some new families of orthogonal polynomials which seem to have nice properties. The asymptotic behavior of the recurrence coefficients may be of interest since in many cases one can observe an oscillating behavior, reminiscent of orthogonal polynomials on two intervals. The paper should also be of interest to people studying discrete Painlevé equations since we are giving two systems of discrete Painlevé equations and a one-parameter family of initial conditions for which the solution turns out to be in terms of recurrence coefficients of orthogonal polynomials on the real line. This means that these particular initial values do not lead to singularities and in fact have nice positivity properties, such as $a_n > 0$. The fact that the polynomials are orthogonal on a (bi-)lattice also gives some properties for partial sums of the b_n (sums of the zeros). The cases where

the bi-lattice reduces to a single lattice may well be corresponding to unique solutions of the discrete Painlevé equations with a prescribed asymptotic behavior for the b_n as $n \rightarrow \infty$.

2 A generalized Charlier weight

2.1 The lattice \mathbb{N}

We consider the discrete weights

$$w_k = \frac{a^k}{(\beta)_k k!}, \quad k \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$$

with $a > 0$. We can write these in terms of the weight function

$$w(x) = \frac{\Gamma(\beta)a^x}{\Gamma(\beta+x)\Gamma(x+1)}, \quad (2.1)$$

which is now a function of $x \in \mathbb{C}$ vanishing whenever x is a pole of $\Gamma(x+1)$ or a pole of $\Gamma(\beta+x)$, i.e., for $x = -1, -2, -3, \dots$ and $x = -\beta, -\beta-1, -\beta-2, \dots$. This weight satisfies the Pearson equation

$$\nabla w(x) := w(x) - w(x-1) = \frac{a - x(\beta-1) - x^2}{a} w(x). \quad (2.2)$$

With this weight we can introduce the inner product

$$\langle f, g \rangle = \sum_{k=0}^{\infty} f(k)g(k)w_k$$

which has positive weights for every $\beta > 0$. We denote the corresponding monic orthogonal polynomials by $P_n(x; a, \beta)$:

$$\sum_{k=0}^{\infty} P_n(k; a, \beta) P_m(k; a, \beta) \frac{a^k}{(\beta)_k k!} = 0, \quad n \neq m. \quad (2.3)$$

Our main interest is to find the recurrence coefficients in the three term recurrence relation

$$xP_n(x; a, \beta) = P_{n+1}(x; a, \beta) + b_n P_n(x; a, \beta) + a_n^2 P_{n-1}(x; a, \beta), \quad (2.4)$$

with initial conditions $P_{-1} = 0$ and $P_0 = 1$. We will obtain recursive relations for these recurrence coefficients. We sometimes just write $P_n(x)$ for the polynomials $P_n(x; a, \beta)$ to simplify the notation. Observe that for $a = \beta c$ and $\beta \rightarrow \infty$ we get the weights $w_k = c^k/k!$ so that

$$\lim_{\beta \rightarrow \infty} P_n(x; \beta c, \beta) = \hat{C}_n(x; c)$$

are the monic Charlier polynomials.

The Pearson equation and summation by parts gives the following structure relation for the orthogonal polynomials

Lemma 1. *The monic orthogonal polynomials given by (2.3) satisfy the relation*

$$\Delta P_n(x) := P_n(x+1) - P_n(x) = nP_{n-1}(x) + B_n P_{n-2}(x), \quad (2.5)$$

for some sequence B_n of real numbers.

Proof. We can always write the polynomial $P_n(x+1) - P_n(x)$ (which is of degree $n-1$) in a Fourier series using the orthogonal polynomials and hence

$$P_n(x+1) - P_n(x) = \sum_{k=0}^{n-1} A_{k,n} P_k(x),$$

where the Fourier coefficients are given by

$$A_{k,n} = \frac{\langle \Delta P_n, P_k \rangle}{\langle P_k, P_k \rangle}.$$

Recall summation by parts: if $b_{-1} = 0$ then

$$\sum_{k=0}^{\infty} (\Delta a_k) b_k = - \sum_{k=0}^{\infty} a_k \nabla b_k$$

whenever a and b are in ℓ_2 . If we apply this, then

$$\begin{aligned} \langle P_k, P_k \rangle A_{k,n} &= \sum_{j=0}^{\infty} (\Delta P_n(j)) P_k(j) w_j \\ &= - \sum_{j=0}^{\infty} P_n(j) \nabla (P_k(j) w_j) \\ &= - \sum_{j=0}^{\infty} P_n(j) w_j \nabla P_k(j) - \sum_{j=0}^{\infty} P_n(j) P_k(j-1) \nabla w_j. \end{aligned}$$

The first sum on the right is zero by orthogonality since ∇P_k is a polynomial of degree $k-1 < n$. For the second sum we use the Pearson equation (2.2) to find

$$\langle P_k, P_k \rangle A_{k,n} = -\frac{1}{a} \sum_{j=0}^{\infty} P_n(j) P_k(j-1) (a - j(\beta - 1) - j^2) w_j.$$

This sum is zero by orthogonality whenever $k+2 < n$. Hence only $A_{n-1,n}$ and $A_{n-2,n}$ can be non-zero. By comparing the leading coefficients, we see that $A_{n-1,n} = n$. If we call $A_{n-2,n} = B_n$, then the required formula follows. \square

Some of these coefficients are useful in later computations. If we define

$$\frac{1}{\gamma_n^2} = \langle P_n, P_n \rangle,$$

then it is not difficult to find from the recurrence relation (2.4) (taking the inner product with P_{n-1}) the well-known relation

$$a_n^2 = \frac{\gamma_{n-1}^2}{\gamma_n^2}.$$

Furthermore, with the notation in the proof of the lemma

$$1 = A_{0,1} = -\frac{\gamma_0^2}{a} \sum_{j=0}^{\infty} P_1(j)(a - j(\beta - 1) - j^2)w_j.$$

By orthogonality we have

$$\sum_{j=0}^{\infty} P_1(j)w_j = 0.$$

If we use the recurrence relation then

$$\sum_{j=0}^{\infty} jP_1(j)w_j = \sum_{j=0}^{\infty} [P_2(j) + b_1P_1(j) + a_1^2P_0(j)]w_j = a_1^2/\gamma_0^2.$$

If we use the recurrence relation twice, then

$$\begin{aligned} \sum_{j=0}^{\infty} j^2P_1(j)w_j &= \sum_{j=0}^{\infty} [b_1(P_2(j) + b_1P_1(j) + a_1^2P_0(j)) + a_1^2(P_1(j) + b_0P_0(j))]w_j \\ &= a_1^2(b_1 + b_0)/\gamma_0^2. \end{aligned}$$

Hence, combining these results, we have

$$1 = \frac{a_1^2}{a}(b_1 + b_0 + \beta - 1). \quad (2.6)$$

In a similar way we can compute B_2 :

$$B_2 = A_{0,2} = -\frac{\gamma_0^2}{a} \sum_{j=0}^{\infty} P_2(j)(a - j(\beta - 1) - j^2)w_j.$$

By orthogonality

$$\sum_{j=0}^{\infty} P_2(j)w_j = 0 = \sum_{j=0}^{\infty} jP_2(j)w_j$$

and using the recurrence relation we get

$$\begin{aligned} \sum_{j=0}^{\infty} j^2P_2(j)w_j &= \sum_{j=0}^{\infty} j(P_3(j) + b_2P_2(j) + a_2^2P_1(j))w_j \\ &= a_2^2a_1^2/\gamma_0^2, \end{aligned}$$

so that

$$B_2 = \frac{a_1^2a_2^2}{a}. \quad (2.7)$$

We now are able to derive the recursive relations for the recurrence coefficients.

Theorem 2.1. *The recurrence coefficients for the orthogonal polynomials defined by (2.3) satisfy*

$$b_n + b_{n-1} - n + \beta = \frac{an}{a_n^2}, \quad (2.8)$$

$$(a_{n+1}^2 - a)(a_n^2 - a) = a(b_n - n)(b_n - n + \beta - 1), \quad (2.9)$$

with initial conditions

$$a_0^2 = 0, \quad b_0 = \frac{\sqrt{a}I_\beta(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})},$$

where I_ν is the modified Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

Proof. On one hand we have the three-term recurrence relation (2.4)

$$kP_n(k) = P_{n+1}(k) + b_nP_n(k) + a_n^2P_{n-1}(k),$$

and on the other hand we have the structure relation (2.5). The compatibility between these two equations will result in the desired equations. Take the forward difference of the three-term recurrence relation to find

$$(k+1)\Delta(P_n(k)) + P_n(k) = \Delta P_{n+1}(k) + b_n\Delta P_n(k) + a_n^2\Delta P_{n-1}(k).$$

Then use the structure relation (2.5) to find

$$\begin{aligned} (k+1)(nP_{n-1}(k) + B_nP_{n-2}(k)) + P_n(k) \\ = (n+1)P_n(k) + B_{n+1}P_{n-1}(k) + b_n(nP_{n-1}(k) + B_nP_{n-2}(k)) \\ + a_n^2((n-1)P_{n-2}(k) + B_{n-1}P_{n-3}(k)). \end{aligned}$$

Finally, use the three-term recurrence relation to work out $kP_{n-1}(k)$ and $kP_{n-2}(k)$ on the left hand side. This gives a linear identity involving $P_n(k), P_{n-1}(k), P_{n-2}(k), P_{n-3}(k)$, and since these four polynomials (in the variable k) are linearly independent, the coefficients should all be zero. This gives the four equations

$$n+1 = n+1, \tag{2.10}$$

$$n + nb_{n-1} + B_n = B_{n+1} + nb_n, \tag{2.11}$$

$$B_n + na_{n-1}^2 + B_nb_{n-2} = B_nb_n + (n-1)a_n^2, \tag{2.12}$$

$$B_na_{n-2}^2 = a_n^2B_{n-1}. \tag{2.13}$$

Clearly (2.10) is always satisfied and (2.13) readily gives

$$\frac{B_n}{a_n^2a_{n-1}^2} = \frac{B_{n-1}}{a_{n-1}^2a_{n-2}^2},$$

so that $B_n = a_n^2a_{n-1}^2B_2/(a_2^2a_1^2)$, and hence (2.7) gives

$$B_n = \frac{a_n^2a_{n-1}^2}{a}, \quad n \geq 2. \tag{2.14}$$

Use this in (2.11) to find

$$na(b_n - b_{n-1} - 1) = a_n^2(a_{n-1}^2 - a_{n+1}^2), \tag{2.15}$$

and (2.12) becomes

$$\frac{1}{a}(b_n - b_{n-2} - 1) = \frac{n}{a_n^2} - \frac{n-1}{a_{n-1}^2}.$$

Summing the latter starting from $n = 2$ gives

$$\frac{1}{a}(b_n + b_{n-1} - n + 1) - \frac{1}{a}(b_1 + b_0) = \frac{n}{a_n^2} - \frac{1}{a_1^2}.$$

Now use (2.6) to find (2.8).

If we use (2.8) in (2.15) and put $b_n = n + d_n$, then we find

$$(d_k + d_{k-1} + k + \beta - 1)(d_{k-1} - d_k) = a_{k+1}^2 - a_{k-1}^2.$$

Summing from $k = 1$ to n gives (use the telescoping property and summation by parts)

$$-d_n^2 + d_0^2 + \sum_{k=0}^{n-1} d_k - nd_n - (\beta - 1)(d_n - d_0) = a_{n+1}^2 + a_n^2 - a_1^2, \quad (2.16)$$

where we used the initial condition $a_0^2 = 0$. On the other hand, (2.15) is equivalent with

$$ak(d_{k-1} - d_k) = a_k^2 a_{k+1}^2 - a_{k-1}^2 a_k^2.$$

Summing from $k = 1$ to n now gives

$$a \sum_{k=0}^{n-1} d_k - and_n = a_n^2 a_{n+1}^2. \quad (2.17)$$

If we use (2.17) in (2.16) then we find

$$-d_n^2 + d_0^2 + \frac{a_n^2 a_{n+1}^2}{a} - (\beta - 1)(d_n - d_0) = a_{n+1}^2 + a_n^2 - a_1^2. \quad (2.18)$$

The initial values $d_0 = b_0$ and a_1^2 are given by

$$b_0 = \frac{m_1}{m_0}, \quad a_1^2 = \frac{m_2}{m_0} - \left(\frac{m_1}{m_0} \right)^2,$$

where m_j are the moments

$$m_j = \sum_{k=0}^{\infty} k^j w_k.$$

A simple calculation gives

$$m_0 = \frac{\Gamma(\beta)}{(\sqrt{a})^{\beta-1}} I_{\beta-1}(2\sqrt{a}), \quad m_1 = \frac{\Gamma(\beta)}{(\sqrt{a})^{\beta-2}} I_{\beta}(2\sqrt{a}),$$

where I_{β} and $I_{\beta-1}$ are modified Bessel functions. This gives

$$b_0 = \sqrt{a} \frac{I_{\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})}. \quad (2.19)$$

The Pearson equation (2.2) gives

$$m_2 = \sum_{k=0}^{\infty} k^2 w_k = \sum_{k=0}^{\infty} [a - k(\beta - 1)] w_k - a \sum_{k=0}^{\infty} [w_k - w_{k-1}],$$

which readily gives $m_2 = am_0 - (\beta - 1)m_1$. Observe that this gives

$$d_0^2 + (\beta - 1)d_0 + a_1^2 = a$$

which simplifies the recurrence relation (2.18) to

$$a_{n+1}^2 + a_n^2 + d_n^2 - \frac{a_n^2 a_{n+1}^2}{a} + (\beta - 1)d_n - a = 0,$$

which is equivalent with (2.9). \square

The case $\beta = 1$ was already considered in [9] (see also [8, §4.2]). Equation (2.9) then simplifies to

$$(a_{n+1}^2 - a)(a_n^2 - a) = (b_n - n)^2, \quad (2.20)$$

so that $a_n^2 - a$ and $a_{n+1}^2 - a$ have the same sign. The boundary condition $a_0^2 = 0$ thus implies that $a_n^2 - a < 0$ for all n , and we can write $a_n^2 - a = -ac_n^2$, for some new sequence $(c_n)_{n \in \mathbb{N}}$. Then $a_n^2 = a(1 - c_n^2)$ so that $c_n^2 < 1$ for $n \geq 1$. This still leaves two choices for the sign of c_n . Taking square roots in (2.20) gives $b_n = n + ac_n c_{n+1}$, where we choose $c_0 = 1$ and we recursively take the sign of c_{n+1} equal to the sign of $(b_n - n)/c_n$. Inserting these formulas for a_n^2 and b_n into (2.8) with $\beta = 1$ gives $a(c_{n+1} + c_{n-1}) = n/(1 - c_n^2)$, which is the discrete Painlevé II equation ([4], [8, Appendix A.1]).

The situation is different for $\beta \neq 1$, since (2.9) now is a quadratic equation in b_n and it is not a priori clear which of the two roots one should choose. Equations (2.8)–(2.9) are a limiting case of a discrete Painlevé IV equation: take the first dP_{IV} in [8, p. 723], or equivalently the second discrete Painlevé equation for D_4^c in [4, p. 297]

$$\begin{aligned} x_{n+1}x_n &= \frac{(y_n - z_n)^2 - A}{y_n^2 - B} \\ y_n + y_{n-1} &= \frac{\zeta_n - C}{1 + Dx_n} + \frac{\zeta_n + C}{1 + x_n/D} \end{aligned}$$

where $z_n = z_0 + n\delta$ and $\zeta_n = z_n - \delta/2$. If we put $x_n = iX_n/\sqrt{aB}$ and let $B \rightarrow \infty$ then this gives for the first equation

$$X_{n+1}X_n = a((y_n - z_n)^2 - A),$$

and if we take $iD = \sqrt{B/a}$ and let $B \rightarrow \infty$ then we also get

$$y_n + y_{n-1} = \frac{\zeta_n - C}{1 + X_n/a} + \zeta_n + C.$$

The parameters $A = (\beta - 1)^2/4$, $C = -\beta/2$, $z_n = n - (\beta - 1)/2$ then give the discrete equations (2.8)–(2.9) for $X_n = a_n^2 - a$ and $y_n = b_n$.

2.2 The shifted lattice $\mathbb{N} + 1 - \beta$

We can consider the weight w in (2.1) also on the shifted lattice $\mathbb{N} + 1 - \beta = \{1 - \beta, 2 - \beta, 3 - \beta, \dots\}$, where

$$v_k := w(k + 1 - \beta) = \frac{\Gamma(\beta)a^{1-\beta}}{\Gamma(2 - \beta)} \frac{a^k}{k!(2 - \beta)_k}, \quad k \in \mathbb{N}.$$

The weights $(v_k)_{k \in \mathbb{N}}$ are therefore of the same form as in the previous section (up to a real factor) but with β replaced by $2 - \beta$. The corresponding monic orthogonal polynomials Q_n on the shifted lattice $\mathbb{N} + 1 - \beta$ satisfy

$$\sum_{k=0}^{\infty} Q_n(k + 1 - \beta) Q_m(k + 1 - \beta) v_k = 0, \quad n \neq m, \quad (2.21)$$

and they are simply the polynomials $P_n(\cdot; a, 2 - \beta)$ but shifted over a distance $1 - \beta$:

$$Q_n(x) = P_n(x + \beta - 1; a, 2 - \beta). \quad (2.22)$$

These are orthogonal polynomials with a positive measure whenever $\beta < 2$. The remarkable fact is that the recurrence coefficients in the three-term recurrence relation

$$xQ_n(x) = Q_{n+1}(x) + \hat{b}_n Q_n(x) + \hat{a}_n^2 Q_{n-1}(x) \quad (2.23)$$

satisfy the same system of non-linear recurrence relations (discrete Painlevé equations) as in Theorem 2.1, but with a different initial condition.

Theorem 2.2. *The recurrence coefficients for the orthogonal polynomials defined by (2.21) satisfy the system of equations*

$$\begin{aligned} \hat{b}_n + \hat{b}_{n-1} - n + \beta &= \frac{an}{\hat{a}_n^2}, \\ (\hat{a}_{n+1}^2 - a)(\hat{a}_n^2 - a) &= a(\hat{b}_n - n)(\hat{b}_n - n + \beta - 1), \end{aligned}$$

with initial conditions

$$\hat{a}_0^2 = 0, \quad \hat{b}_0 = \frac{\sqrt{a} I_{-\beta}(2\sqrt{a})}{I_{1-\beta}(2\sqrt{a})},$$

where I_ν is the modified Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k! \Gamma(k + \nu + 1)}.$$

Proof. We will give two ways to prove the result. The first way is to use the relation (2.22), which readily gives $\hat{a}_n^2 = a_n^2$ and $\hat{b}_n = b_n + 1 - \beta$, where a_n and b_n are the recurrence coefficients for the polynomials $P_n(x; a, 2 - \beta)$. We can therefore use Theorem 2.1, but with β replaced by $2 - \beta$, and substitute a_n^2 by \hat{a}_n^2 and b_n by $\hat{b}_n - 1 + \beta$. This indeed leaves the equations unchanged. The initial conditions are $\hat{a}_0^2 = 0$ and

$$\hat{b}_0 = b_0 + 1 - \beta = \frac{\sqrt{a} I_{2-\beta}(2\sqrt{a}) + (1 - \beta) I_{1-\beta}(2\sqrt{a})}{I_{1-\beta}(2\sqrt{a})}.$$

The latter expression can be simplified by using a well-known recurrence relation for modified Bessel functions [7, Eq. 10.29.1 on p. 251].

An alternative way is to repeat the proof of Theorem 2.1, but now on the lattice $\mathbb{N} + 1 - \beta$. Observe that we only used the Pearson equation (2.2) and the boundary condition $w(-1) = 0$ in the proof of Theorem 2.1 to arrive at the discrete equations (2.8)–(2.9). In the present case the Pearson equation is still valid and we now use the

boundary condition $w(-\beta) = 0$, which allows us to use summation by parts without the boundary terms. The only difference is that the moments are now

$$\begin{aligned}\hat{m}_0 &= \sum_{k=0}^{\infty} w(k+1-\beta) \\ &= \Gamma(\beta) a^{1-\beta} \sum_{k=0}^{\infty} \frac{a^k}{k! \Gamma(k+2-\beta)} = \Gamma(\beta) (\sqrt{a})^{1-\beta} I_{1-\beta}(2\sqrt{a}), \\ \hat{m}_1 &= \sum_{k=0}^{\infty} (k+1-\beta) w(k+1-\beta) \\ &= \Gamma(\beta) a^{1-\beta} \sum_{k=0}^{\infty} \frac{a^k}{k! \Gamma(k+1-\beta)} = \Gamma(\beta) (\sqrt{a})^{2-\beta} I_{-\beta}(2\sqrt{a}),\end{aligned}$$

so that $\hat{b}_0 = \hat{m}_1/\hat{m}_0 = \sqrt{a} I_{-\beta}(2\sqrt{a})/I_{1-\beta}(2\sqrt{a})$. \square

2.3 Combining both lattices

Now we can combine the two lattices and study orthogonal polynomials on the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$. We use the orthogonality measure $\mu = c_1 \mu_1 + c_2 \mu_2$, where $c_1, c_2 > 0$, μ_1 is the discrete measure on \mathbb{N} with weights $w_k = w(k)$, and μ_2 is the discrete measure on $\mathbb{N} + 1 - \beta$ with weights $v_k = w(k+1-\beta)$. This discrete measure depends on two parameters c_1, c_2 , but the orthogonal polynomials will only depend on their ratio $t = c_2/c_1 > 0$. Let $0 < \beta < 2$, then μ is a positive measure and the monic orthogonal polynomials $R_n(x) = R_n(x; a, \beta, t)$ satisfy the three-term recurrence relation

$$x R_n(x) = R_{n+1}(x) + \tilde{b}_n R_n(x) + \tilde{a}_n^2 R_{n-1}(x). \quad (2.24)$$

It is remarkable that these recurrence coefficients again satisfy the same non-linear recurrence relations (discrete Painlevé equations) as in Theorem 2.1 and Theorem 2.2, but with an initial condition depending on the parameter $t = c_2/c_1$.

Theorem 2.3. *The recurrence coefficients for the orthogonal polynomials defined by*

$$c_1 \sum_{k=0}^{\infty} R_n(k) R_m(k) w_k + c_2 \sum_{k=0}^{\infty} R_n(k+1-\beta) R_m(k+1-\beta) v_k = 0, \quad m \neq n, \quad (2.25)$$

satisfy the system of equations

$$\begin{aligned}\tilde{b}_n + \tilde{b}_{n-1} - n + \beta &= \frac{an}{\tilde{a}_n^2}, \\ (\tilde{a}_{n+1}^2 - a)(\tilde{a}_n^2 - a) &= a(\tilde{b}_n - n)(\tilde{b}_n - n + \beta - 1),\end{aligned}$$

with initial conditions

$$\tilde{a}_0^2 = 0, \quad \tilde{b}_0 = \sqrt{a} \frac{I_{\beta}(2\sqrt{a}) + t I_{-\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a}) + t I_{1-\beta}(2\sqrt{a})},$$

where I_{ν} is the modified Bessel function

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k! \Gamma(k+\nu+1)}.$$

Proof. Going through the proof of Theorem 2.1 we observe that only the Pearson equation (2.2) is used, together with summation by parts. The boundary conditions $w(-1) = 0$ and $w(-\beta) = 0$ ensure that this summation by parts does not leave any boundary terms to evaluate at -1 (for the first sum) or $-\beta$ (for the second sum). Hence the recurrence coefficients will satisfy the same non-linear recurrence relations as in Theorem 2.1 and 2.2. The only difference is that the initial condition for $\tilde{b}_0 = \tilde{m}_1/\tilde{m}_0$ now is in terms of the moments

$$\tilde{m}_0 = c_1 m_0 + c_2 \hat{m}_0 = \Gamma(\beta)(\sqrt{a})^{1-\beta} (c_1 I_{\beta-1}(2\sqrt{a}) + c_2 I_{1-\beta}(2\sqrt{a})),$$

and

$$\tilde{m}_1 = c_1 m_0 + c_2 \hat{m}_0 = \Gamma(\beta)(\sqrt{a})^{2-\beta} (c_1 I_{\beta}(2\sqrt{a}) + c_2 I_{-\beta}(2\sqrt{a})).$$

□

We have now identified special solutions of the discrete system (2.8)–(2.9) with initial value $a_0^2 = 0$ and

$$b_0(t) = \sqrt{a} \frac{I_{\beta}(2\sqrt{a}) + t I_{-\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a}) + t I_{1-\beta}(2\sqrt{a})}$$

which depends on one parameter $t \in (0, \infty)$. If we use the relation [7, Eq. 10.27.2 on p. 251]

$$I_{-\nu}(z) = I_{\nu}(z) + \frac{2}{\pi} \sin \nu \pi K_{\nu}(z),$$

where K_{ν} is the other modified Bessel function, then the initial condition can also be written as

$$b_0 = \sqrt{a} \frac{I_{\beta}(2\sqrt{a}) + s K_{\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a}) - s K_{\beta-1}(2\sqrt{a})}, \quad s = \frac{2t}{1+t} \frac{\sin \beta \pi}{\pi}.$$

Observe that

$$\begin{aligned} \frac{I_{\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})} &< \frac{I_{-\beta}(2\sqrt{a})}{I_{1-\beta}(2\sqrt{a})}, & \text{if } 0 < \beta < 1, \\ \frac{I_{\beta}(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})} &> \frac{I_{-\beta}(2\sqrt{a})}{I_{1-\beta}(2\sqrt{a})}, & \text{if } 1 < \beta < 2, \end{aligned}$$

so that $b_0(t)$ is a monotonically increasing function of $t \in [0, \infty]$ when $0 < \beta < 1$ and a monotonically decreasing function when $1 < \beta < 2$. One can use the Wronskian formula [7, Eq. 10.28.1 on p. 251] and the product formula [7, Eq. 10.32.15 on p. 253] to verify this. For each initial value in $[b_0(0), b_0(\infty)]$ (when $0 < \beta < 1$) or $[b_0(\infty), b_0(0)]$ (when $1 < \beta < 2$) the solution therefore corresponds to recurrence coefficients of orthogonal polynomials with a positive measure on the real line (in fact on the bi-lattice $\mathbb{N} \cup (\mathbb{N}+1-\beta)$) whenever $a > 0$, and hence this solution satisfies certain positivity constraints: $a_n^2 > 0$ for $n \geq 1$ and $b_n > \min(0, 1-\beta)$ for all $n \geq 0$. Moreover, since the orthogonal polynomials are on a discrete set, we can use the familiar fact for orthogonal polynomials that between two zeros there is at least one point of the bi-lattice [3, Theorem 4.1 on p. 59]. The sum of the zeros $x_{1,n} < x_{2,n} < \dots < x_{n,n}$ of R_n hence satisfies

$$\sum_{k=1}^n x_{k,n} > \sum_{k=0}^{n-1} y_k$$

where $y_0 < y_1 < y_2 < \dots$ are the points in the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$, i.e.,

$$\begin{cases} y_{2k} = k, & y_{2k+1} = k + 1 - \beta, & \text{if } 0 < \beta < 1, \\ y_{2k} = k + 1 - \beta, & y_{2k+1} = k, & \text{if } 1 < \beta < 2, \\ y_k = k, & & \text{if } \beta = 1. \end{cases}$$

The sum of the zeros is the trace of the truncated Jacobi matrix containing the recurrence coefficients, hence

$$\sum_{k=1}^n x_{k,n} = \sum_{k=0}^{n-1} b_k,$$

so that we get the constraint

$$\begin{aligned} \sum_{k=0}^{2n-1} b_k &> n(n-1) + n(1-\beta), & \text{if } 0 < \beta < 2 \text{ and } \beta \neq 1, \\ \sum_{k=0}^{n-1} b_k &> n(n-1)/2, & \text{if } \beta = 1. \end{aligned}$$

The sum $\sum_{k=0}^{n-1} b_k$ behaves like $n^2/4 + \mathcal{O}(n)$ as $n \rightarrow \infty$ when $\beta \neq 1$ and as $n^2/2 + \mathcal{O}(n)$ when $\beta = 1$. This result follows from [6, Theorem 2.2] (see in particular the example of Charlier polynomials on p. 200). This difference in the behavior of the recurrence coefficients is easily explained since for $\beta = 1$ the two lattices coincide and we just have one lattice \mathbb{N} . A similar situation appears when $t \rightarrow 0$ (which gives the orthogonal polynomials P_n on the lattice \mathbb{N}) and when $t \rightarrow \infty$, which gives the polynomials Q_n on the lattice $\mathbb{N} + 1 - \beta$. We believe that these extreme cases ($t = 0$, $t = \infty$ and $\beta = 1$) correspond to the only solution of (2.8)–(2.9) with $a_0^2 = 0$ for which $a_n^2 > 0$ for all $n > 0$ and $b_n = n + \mathcal{O}(1)$ as $n \rightarrow \infty$.

Another special case occurs when $\beta = 1/2$. In this case the bi-lattice is equally spaced and equal to $\frac{1}{2}\mathbb{N}$. The lattice points are $y_k = k/2$ and the weight w at these points is

$$w(k/2) = \frac{\sqrt{\pi} a^{k/2}}{\Gamma((k+1)/2) \Gamma(k/2+1)} = \frac{2^k a^{k/2}}{\Gamma(k+1)}$$

where we used Legendre's duplication formula [7, Eq. 5.5.5 on p. 138] for the gamma function. This means that if we take $c_1 = c_2$, then the weights of the measure $\mu = \mu_1 + \mu_2$ are precisely the Charlier weights (Poisson distribution) with parameter $2\sqrt{a}$. The orthogonal polynomials are therefore Charlier polynomials [3, Chapter VI, §1] but with a scaling:

$$R_n(x; a, 1/2, 1) = 2^{-n} \hat{C}_n(2x; 2\sqrt{a}),$$

where $\hat{C}_n(x; a)$ are the monic Charlier polynomials with parameter a . The recurrence coefficients of Charlier polynomials $\hat{C}_n(x; a)$ are known to be $a_n^2 = na$ and $b_n = n + a$, so for the scaled polynomials $2^{-n} \hat{C}_n(2x; a)$ they are $a_n^2 = na/4$ and $b_n = (n + a)/2$, and for our polynomials $R_n(x) = 2^{-n} \hat{C}_n(2x; 2\sqrt{a})$ we thus have

$$\tilde{a}_n^2 = \frac{n\sqrt{a}}{2}, \quad \tilde{b}_n = \frac{n}{2} + \sqrt{a}. \quad (2.26)$$

One can indeed verify that (2.26) is a solution of (2.8)–(2.9) with initial condition

$$\tilde{b}_0 = \sqrt{a} \frac{I_{1/2}(2\sqrt{a}) + I_{-1/2}(2\sqrt{a})}{I_{-1/2}(2\sqrt{a}) + I_{1/2}(2\sqrt{a})} = \sqrt{a}.$$

So there are special values of β for which the non-linear equations have simple solutions (as a function of n).

3 A generalized Meixner weight

3.1 The lattice \mathbb{N}

We consider the sequence $(p_n)_n$ of polynomials, *orthonormal* with respect to the weights

$$w_k = \frac{(\gamma)_k a^k}{k! (\beta)_k}, \quad k \in \mathbb{N} = \{0, 1, 2, 3, \dots\}. \quad (3.1)$$

These polynomials satisfy

$$\sum_{k=0}^{\infty} p_n(k) p_m(k) w_k = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta. These weights are positive whenever $\beta, \gamma, a > 0$. Notice that the special case $\beta = 1$ was studied in [1], where the authors showed that the recurrence coefficients a_n, b_n satisfy a limiting case of the asymmetric discrete Painlevé equation α -dP_{IV}. The special case $\beta = \gamma$ gives the well-known Charlier weight. We can write these weights in terms of a weight function w given by

$$w(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \frac{\Gamma(\gamma + x) a^x}{\Gamma(\beta + x) \Gamma(x + 1)} \quad (3.2)$$

so that $w_k = w(k)$ for $k \in \mathbb{N}$.

We will prove the following result on the recurrence coefficients of the sequence of orthonormal polynomials.

Theorem 3.1. *The orthonormal polynomials with respect to the weights (3.1) on the lattice \mathbb{N} , with $\gamma, \beta, a > 0$, satisfy the three-term recurrence relation*

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where the recurrence coefficients are given by the initial conditions

$$a_0 = 0, \quad b_0 = \frac{\gamma a}{\beta} \frac{M(\gamma + 1, \beta + 1, a)}{M(\gamma, \beta, a)},$$

where $M(a, b, z)$ is the confluent hypergeometric function

$$M(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k = {}_1F_1(a; b; z),$$

and $a_n^2 = na - (\gamma - 1)u_n$, $b_n = n + \gamma - \beta + a - (\gamma - 1)v_n/a$, where $(u_n, v_n)_{n \in \mathbb{N}}$ satisfy the system of non-linear equations

$$(u_n + v_n)(u_{n+1} + v_n) = \frac{\gamma - 1}{a^2} v_n(v_n - a) \left(v_n - a \frac{\gamma - \beta}{\gamma - 1} \right), \quad (3.3)$$

$$(u_n + v_n)(u_n + v_{n-1}) = \frac{u_n}{u_n - \frac{an}{\gamma - 1}} (u_n + a) \left(u_n + a \frac{\gamma - \beta}{\gamma - 1} \right). \quad (3.4)$$

Proof. We will use the technique of ladder operators, proposed by Chen and Ismail [2]. The ladder operators are defined by

$$\begin{aligned} A_n(x) &= a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_n(\ell - 1) \frac{u(x+1) - u(\ell)}{x+1-\ell} w(\ell), \\ B_n(x) &= a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_{n-1}(\ell - 1) \frac{u(x+1) - u(\ell)}{x+1-\ell} w(\ell), \end{aligned}$$

where $u(x) = -1 + \frac{w(x-1)}{w(x)}$. These ladder operators satisfy

$$A_n(x) p_{n-1}(x) - B_n(x) p_n(x) = p_n(x+1) - p_n(x). \quad (3.5)$$

This can be seen by simplifying the left hand side using the definitions for A_n , B_n and u , and the Christoffel-Darboux identity

$$\sum_{j=0}^n p_j(x) p_j(y) = a_{n+1} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y}.$$

After doing this we obtain

$$A_n(x) p_{n-1}(x) - B_n(x) p_n(x) = \sum_{j=0}^{n-1} p_j(x) \sum_{\ell=0}^{\infty} p_n(\ell+1) p_j(\ell) w(\ell).$$

Exactly the same expression is found by writing $p_n(x+1) - p_n(x)$ as a Fourier series:

$$p_n(x+1) - p_n(x) = \sum_{j=0}^{n-1} \alpha_{n,j} p_j(x).$$

Equation (3.5) gives rise to the compatibility relations

$$B_n(x) + B_{n+1}(x) = \frac{x - b_n}{a_n} A_n(x) - u(x+1) + \sum_{j=0}^n \frac{A_j(x)}{a_j} \quad (3.6)$$

and

$$a_{n+1} A_{n+1}(x) - a_n^2 \frac{A_{n-1}(x)}{a_{n-1}} = (x - b_n) B_{n+1}(x) - (x+1 - b_n) B_n(x) + 1. \quad (3.7)$$

An easy calculation gives that

$$\frac{u(x+1) - u(\ell)}{x+1-\ell} = \frac{1}{a(\gamma + \ell - 1)} \left(\ell + \frac{(\gamma - 1)(x + \beta)}{\gamma + x} \right).$$

Inserting this into the ladder operators, we find that

$$A_n(x) = \frac{a_n}{a} R_n + \frac{a_n}{a} \frac{x + \beta}{x + \gamma} T_n \quad (3.8)$$

and

$$B_n(x) = \frac{1}{a} r_n + \frac{1}{a} \frac{x + \beta}{x + \gamma} t_n, \quad (3.9)$$

where

$$\begin{aligned} R_n &= \sum_{\ell=0}^{\infty} p_n(\ell) p_n(\ell-1) \frac{\ell}{\gamma + \ell - 1} w(\ell), \\ T_n &= \sum_{\ell=0}^{\infty} p_n(\ell) p_n(\ell-1) \frac{\gamma - 1}{\gamma + \ell - 1} w(\ell), \\ r_n &= a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_{n-1}(\ell-1) \frac{\ell}{\gamma + \ell - 1} w(\ell), \\ t_n &= a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_{n-1}(\ell-1) \frac{\gamma - 1}{\gamma + \ell - 1} w(\ell). \end{aligned}$$

We use the expressions (3.8)–(3.9) in the compatibility relations, and compare the coefficients of x^2 , x and 1. In this way we obtain a system of six equations:

$$0 = R_n + T_n - 1, \quad (3.10)$$

$$r_n + t_n + r_{n+1} + t_{n+1} = \gamma R_n - b_n R_n - b_n T_n + \beta T_n + a - \beta - 1 + \sum_{j=0}^n R_j + \sum_{j=0}^n T_j, \quad (3.11)$$

$$\gamma r_n + \beta t_n + \gamma r_{n+1} + \beta t_{n+1} = -b_n \gamma R_n - b_n \beta T_n + \gamma a - \beta + \gamma \sum_{j=0}^n R_j + \beta \sum_{j=0}^n T_j, \quad (3.12)$$

$$0 = r_{n+1} + t_{n+1} - r_n - t_n, \quad (3.13)$$

$$\begin{aligned} &a_{n+1}^2 R_{n+1} + a_{n+1}^2 T_{n+1} - a_n^2 R_{n-1} - a_n^2 T_{n-1} \\ &= (\gamma - b_n) r_{n+1} + (\beta - b_n) t_{n+1} - (1 - b_n + \gamma) r_n - (1 - b_n + \beta) t_n + a, \end{aligned} \quad (3.14)$$

$$\begin{aligned} &\gamma a_{n+1}^2 R_{n+1} + \beta a_{n+1}^2 T_{n+1} - \gamma a_n^2 R_{n-1} - \beta a_n^2 T_{n-1} \\ &= -\gamma b_n r_{n+1} - \beta b_n t_{n+1} - \gamma(1 - b_n) r_n - \beta(1 - b_n) t_n + \gamma a. \end{aligned} \quad (3.15)$$

By (3.10) we can substitute R_n by $1 - T_n$, and by (3.13) (and since $r_0 = t_0 = 0$ by definition) we can substitute r_n by $-t_n$. If we apply these substitutions, (3.11) gives an expression for b_n as a function of T_n :

$$b_n = \gamma - (\gamma - \beta) T_n + a + n - \beta, \quad (3.16)$$

and (3.12) gives

$$(\gamma - \beta)(t_n + t_{n+1}) = \gamma b_n - (\gamma - \beta) b_n T_n - \gamma a + \beta - \gamma(n+1) + (\gamma - \beta) \sum_{j=0}^n T_j.$$

Substituting the newly found expression for b_n in this equation, we obtain

$$t_n + t_{n+1} = (\gamma - \beta)T_n^2 - (2\gamma + a + n - \beta - 1)T_n + \gamma - 1 + \sum_{j=0}^{n-1} T_j. \quad (3.17)$$

On the other hand, elimination of R_n and r_n in (3.15) gives

$$\gamma(a_{n+1}^2 - a_n^2) - (\gamma - \beta)a_{n+1}^2 T_{n+1} + (\gamma - \beta)a_n^2 T_{n-1} = (\gamma - \beta)b_n t_{n+1} + (\gamma - \beta)(1 - b_n)t_n + \gamma a \quad (3.18)$$

and in (3.14) it gives

$$a_{n+1}^2 - a_n^2 = (\gamma - \beta)(t_n - t_{n+1}) + a, \quad (3.19)$$

which after taking a telescoping sum gives us an expression for the recurrence coefficients a_n as a function of t_n :

$$a_n^2 = na - (\gamma - \beta)t_n. \quad (3.20)$$

Notice that since $a_n \geq 0$ for all n , the knowledge of t_n will imply the knowledge of a_n . Using (3.19) and the expression for b_n in (3.18), we obtain

$$a_n^2 T_{n-1} - a_{n+1}^2 T_{n+1} = (b_n + \gamma)t_{n+1} + (1 - b_n - \gamma)t_n, \quad (3.21)$$

in which substitution of (3.20) gives us another relation between terms of the sequences $(t_n)_n$ and $(T_n)_n$:

$$\begin{aligned} -a(n+1)T_{n+1} + anT_{n-1} &= t_{n+1}(n+a+2\gamma-\beta) - t_n(n+a+2\gamma-\beta-1) \\ &\quad - t_{n+1}(\gamma-\beta)(T_n+T_{n+1}) + t_n(\gamma-\beta)(T_n+T_{n-1}). \end{aligned}$$

Taking a telescopic sum, we get

$$(T_n + T_{n-1})(an - (\gamma - \beta)t_n) = -(a + 2\gamma + n - \beta - 1)t_n + a \sum_{j=0}^{n-1} T_j. \quad (3.22)$$

On the other hand, if we multiply (3.21) by T_n and we use (3.16) and (3.17), we obtain

$$a_{n+1}^2 T_{n+1} T_n - a_n^2 T_n T_{n-1} = t_{n+1}^2 - t_n^2 - (\gamma - 1)(t_{n+1} - t_n) - t_{n+1} \sum_{j=0}^n T_j + t_n \sum_{j=0}^{n-1} T_j,$$

which by taking a telescoping sum gives

$$a_n^2 T_n T_{n-1} = t_n \left(t_n - \gamma + 1 - \sum_{j=0}^{n-1} T_j \right). \quad (3.23)$$

If we multiply (3.22) by T_n and (3.17) by t_n , comparison of these two expressions gives

$$anT_n^2 - t_n t_{n+1} = aT_n \sum_{j=0}^{n-1} T_j.$$

Using in this equality the expression for $\sum_{j=0}^{n-1} T_j$ from (3.17), we obtain a relation between T_n , t_n and t_{n+1} :

$$aT_n(T_n - 1)(T_n(\gamma - \beta) - \gamma + 1) = (t_n + aT_n)(t_{n+1} + aT_n). \quad (3.24)$$

Finally, multiplying (3.22) by at_n , and using once again (3.23) to eliminate the sum of T_j , and (3.20) to eliminate a_n^2 , we obtain a relation between T_n , T_{n-1} and t_n :

$$(aT_n + t_n)(aT_{n-1} + t_n) = \frac{t_n(a^2(\gamma - 1) + a(2\gamma - \beta - 1)t_n + (\gamma - \beta)t_n^2)}{(\gamma - \beta)t_n - an}. \quad (3.25)$$

The substitution

$$u_n = \frac{\gamma - \beta}{\gamma - 1}t_n, \quad v_n = \frac{\gamma - \beta}{\gamma - 1}aT_n,$$

allows us to write the system of two equations in a more symmetrical form:

$$\begin{aligned} (u_n + v_n)(u_{n+1} + v_n) &= \frac{\gamma - 1}{a^2}v_n(v_n - a) \left(v_n - a\frac{\gamma - \beta}{\gamma - 1} \right), \\ (u_n + v_n)(u_n + v_{n-1}) &= \frac{u_n}{u_n - \frac{an}{\gamma - 1}}(u_n + a) \left(u_n + a\frac{\gamma - \beta}{\gamma - 1} \right), \end{aligned}$$

which is the system (3.3)–(3.4) given in the Theorem. The initial conditions are given by $u_0 = 0$, since $t_0 = 0$, and $v_0 = \frac{a}{\gamma - 1}(\gamma - \beta + a - \frac{m_1}{m_0})$ since T_0 and b_0 are related by (3.16), and since $b_0 = m_1/m_0$, where m_k is the k -th moment of the weights $(w_k)_{k \in \mathbb{N}}$. The first two moments are easily recognized as confluent hypergeometric functions:

$$m_0 = \sum_{k=0}^{\infty} w_k = M(\gamma, \beta, a)$$

and

$$m_1 = \sum_{k=0}^{\infty} kw_k = \frac{\gamma a}{\beta} M(\gamma + 1, \beta + 1, a).$$

□

The system (3.3)–(3.4) can be seen as a limiting case of the asymmetric discrete Painlevé-IV equation α -dP_{IV} ([4, E_6^δ on p. 296], [8, Appendix A.3]) given by

$$\begin{aligned} (X_n + Y_n)(X_{n+1} + Y_n) &= \frac{(Y_n - A)(Y_n - B)(Y_n - C)(Y_n - D)}{(Y_n + \Gamma - Z_n)(Y_n - \Gamma - Z_n)}, \\ (X_n + Y_n)(X_n + Y_{n-1}) &= \frac{(X_n + A)(X_n + B)(X_n + C)(X_n + D)}{(X_n + \Delta - Z_{n+1/2})(X_n - \Delta - Z_{n+1/2})}, \end{aligned}$$

where $A + B + C + D = 0$ and Z_n is a polynomial of degree one in n . Indeed, let

$$X_n = u_n - \frac{1}{\epsilon}, \quad Y_n = v_n + \frac{1}{\epsilon}, \quad Z_n = \frac{a}{\gamma - 1} \left(n - \frac{1}{2} \right) + \frac{1}{\epsilon},$$

and

$$\begin{aligned} A &= \frac{1}{\epsilon}, \quad B = -\frac{3}{\epsilon} - a - a\frac{\gamma - \beta}{\gamma - 1}, \quad C = a + \frac{1}{\epsilon}, \quad D = \frac{1}{\epsilon} + a\frac{\gamma - \beta}{\gamma - 1}, \\ \Gamma^2 &= \frac{-4a^2}{(\gamma - 1)\epsilon}, \quad \Delta = \frac{2}{\epsilon}, \end{aligned}$$

then letting ϵ tend to zero gives the system (3.3)–(3.4).

3.2 The shifted lattice $\mathbb{N} + 1 - \beta$

The weight function w has zeros at the negative integers $-1, -2, \dots$, but also at $-\beta, -1 - \beta, \dots$. Therefore it makes sense to consider the weight function on the shifted lattice $\mathbb{N} + 1 - \beta$. It is easy to verify that the weight on this shifted lattice is, up to a constant factor, equal to the weight on the original lattice \mathbb{N} , with different parameters. Indeed, if we denote

$$w_{\gamma, \beta, a}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \frac{\Gamma(\gamma + x)a^x}{\Gamma(x + 1)\Gamma(\beta + x)},$$

then

$$w_{\gamma, \beta, a}(k + 1 - \beta) = a^{1-\beta} \frac{\Gamma(\beta)\Gamma(\gamma + 1 - \beta)}{\Gamma(2 - \beta)\Gamma(\gamma)} w_{\gamma+1-\beta, 2-\beta, a}(k).$$

Hence the corresponding orthonormal polynomials q_n , which satisfy

$$\sum_{k=0}^{\infty} q_n(k + 1 - \beta) q_m(k + 1 - \beta) w(k + 1 - \beta) = \delta_{n,m}, \quad (3.26)$$

are, up to a constant factor, equal to the polynomials p_n , shifted in both the variable x and the parameters γ and β :

$$q_n^{\gamma, \beta, a}(x) = \zeta_n p_n^{\gamma+1-\beta, 2-\beta, a}(x + \beta - 1). \quad (3.27)$$

The weights $(w(k + 1 - \beta))_{k \in \mathbb{N}}$ are positive when $a > 0$, $\gamma + 1 - \beta > 0$ and $2 - \beta > 0$, i.e., whenever $a > 0$, $\beta < 2$ and $\gamma > \beta - 1$. Remarkably, the recurrence coefficients in the recurrence relation

$$xq_n(x) = \hat{a}_{n+1}q_{n+1}(x) + \hat{b}_nq_n(x) + \hat{a}_nq_{n-1}(x)$$

satisfy exactly the same system of difference equations as the polynomials p_n , only the initial condition for \hat{b}_0 changes.

Theorem 3.2. *The recurrence coefficients for the orthonormal polynomials q_n , defined by (3.26) are given by $\hat{a}_n^2 = na - (\gamma - 1)\hat{u}_n$, $\hat{b}_n = n + \gamma - \beta + a - (\gamma - 1)\hat{v}_n/a$, where $(\hat{u}_n, \hat{v}_n)_{n \in \mathbb{N}}$ satisfy the system of equations*

$$\begin{aligned} (\hat{u}_n + \hat{v}_n)(\hat{u}_{n+1} + \hat{v}_n) &= \frac{\gamma - 1}{a^2} \hat{v}_n(\hat{v}_n - a) \left(\hat{v}_n - a \frac{\gamma - \beta}{\gamma - 1} \right) \\ (\hat{u}_n + \hat{v}_n)(\hat{u}_n + \hat{v}_{n-1}) &= \frac{\hat{u}_n}{\hat{u}_n - \frac{an}{\gamma-1}} (\hat{u}_n + a) \left(\hat{u}_n + a \frac{\gamma - \beta}{\gamma - 1} \right), \end{aligned}$$

with initial conditions

$$\hat{a}_0 = 0, \quad \hat{b}_0 = (1 - \beta) \frac{M(\gamma - \beta + 1, 1 - \beta, a)}{M(\gamma - \beta + 1, 2 - \beta, a)},$$

where $M(a, b, z)$ is the confluent hypergeometric function.

Proof. The proof of Theorem 3.1 can easily be adapted to this case, where the ladder operators are now given by

$$\begin{aligned}\hat{A}_n(x) &= \hat{a}_n \sum_{\ell=0}^{\infty} q_n(\ell+1-\beta) q_n(\ell-\beta) \frac{u(x+1) - u(\ell+1-\beta)}{x-\ell+\beta} w(\ell+1-\beta), \\ \hat{B}_n(x) &= \hat{a}_n \sum_{\ell=0}^{\infty} q_n(\ell+1-\beta) q_{n-1}(\ell-\beta) \frac{u(x+1) - u(\ell+1-\beta)}{x-\ell+\beta} w(\ell+1-\beta).\end{aligned}$$

They satisfy

$$\hat{A}_n(x) q_{n-1}(x) - \hat{B}_n(x) q_n(x) = q_n(x+1) - q_n(x),$$

leading to exactly the same compatibility relations for \hat{A}_n and \hat{B}_n as in (3.6)–(3.7). Hence we obtain the same difference relation for the recurrence coefficients \hat{a}_n and \hat{b}_n as in the case of the lattice \mathbb{N} . The only difference is the initial condition for \hat{b}_0 , which can again be obtained by calculating \hat{m}_1/\hat{m}_0 .

Alternatively, if we use the relation (3.27), then we see that

$$\hat{a}_n^2 = a_n^2(\gamma - \beta + 1, 2 - \beta, a), \quad \hat{b}_n = b_n(\gamma - \beta + 1, 2 - \beta, a) + 1 - \beta,$$

where a_n and b_n are the recurrence coefficients of the polynomials in Theorem 3.1 but with γ replaced by $\gamma - \beta + 1$ and β replaced by $2 - \beta$. This means that

$$\hat{u}_n = \frac{\gamma - \beta}{\gamma - 1} u_n(\gamma - \beta + 1, 2 - \beta, a), \quad \hat{v}_n = \frac{\gamma - \beta}{\gamma - 1} v_n(\gamma - \beta + 1, 2 - \beta, a)$$

and if we insert this in (3.3)–(3.4), then we retrieve the same non-linear system of equations for $(\hat{u}_n, \hat{v}_n)_{n \in \mathbb{N}}$. The initial condition is

$$\hat{b}_0 = b_0(\gamma - \beta + 1, 2 - \beta, a) + 1 - \beta,$$

which becomes

$$\hat{b}_0 = \frac{a(\gamma - \beta + 1)M(\gamma + 2 - \beta, 3 - \beta, a) + (2 - \beta)(1 - \beta)M(\gamma + 1 - \beta, 2 - \beta, a)}{(2 - \beta)M(\gamma - \beta + 1, 2 - \beta, a)}.$$

This can be simplified by using some recurrence relations for confluent hypergeometric functions, in particular 13.3.3 and 13.3.4 in [7, p. 325], which leads to

$$\hat{b}_0 = (1 - \beta) \frac{M(\gamma - \beta + 1, 1 - \beta, a)}{M(\gamma - \beta + 1, 2 - \beta, a)}.$$

□

3.3 Combining both lattices

We can now consider a combination of both lattices by considering the measure $\mu = \mu_1 + t\mu_2$ on the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$, where μ_1 is the discrete measure on \mathbb{N} with weights $(w(k))_{k \in \mathbb{N}}$, and μ_2 is the discrete measure on $\mathbb{N} + 1 - \beta$ with weights $(w(k + 1 - \beta))_{k \in \mathbb{N}}$. In order to have two positive measures μ_1 and μ_2 we impose the conditions

$$a > 0, \quad 0 < \beta < 2, \quad \gamma > \max(0, \beta - 1), \quad t \in [0, \infty].$$

The orthonormal polynomials $(r_n)_{n \in \mathbb{N}}$ for this measure depend on $t > 0$. Once again the recurrence coefficients which appear in the recurrence relation

$$xr_n(x) = \tilde{a}_{n+1}r_{n+1}(x) + \tilde{b}_nr_n(x) + \tilde{a}_nr_{n-1}(x)$$

satisfy the same system of non-linear equations.

Theorem 3.3. *The recurrence coefficients for the orthonormal polynomials r_n , defined by*

$$\sum_{k=0}^{\infty} r_n(k)r_m(k)w(k) + t \sum_{k=0}^{\infty} r_n(k+1-\beta)r_m(k+1-\beta)w(k+1-\beta) = \delta_{n,m}$$

are given by $\tilde{a}_n^2 = na - (\gamma - 1)\tilde{u}_n$ and $\tilde{b}_n = n + \gamma - \beta + a - (\gamma - 1)\tilde{v}_n/a$, where $(\tilde{u}_n, \tilde{v}_n)_{n \in \mathbb{N}}$ satisfy the system of non-linear equations

$$\begin{aligned} (\tilde{u}_n + \tilde{v}_n)(\tilde{u}_{n+1} + \tilde{v}_n) &= \frac{\gamma - 1}{a^2} \tilde{v}_n(\tilde{v}_n - a) \left(\tilde{v}_n - a \frac{\gamma - \beta}{\gamma - 1} \right), \\ (\tilde{u}_n + \tilde{v}_n)(\tilde{u}_n + \tilde{v}_{n-1}) &= \frac{\tilde{u}_n}{\tilde{u}_n - \frac{an}{\gamma - 1}} (\tilde{u}_n + a) \left(\tilde{u}_n + a \frac{\gamma - \beta}{\gamma - 1} \right), \end{aligned}$$

with initial conditions $\tilde{a}_0 = 0$ and

$$\tilde{b}_0 = \frac{m_1 + t\hat{m}_1}{m_0 + t\hat{m}_0}, \quad (3.28)$$

where

$$\begin{aligned} m_0 &= M(\gamma, \beta, a), \quad m_1 = \frac{\gamma a}{\beta} M(\gamma + 1, \beta + 1, a), \\ \hat{m}_0 &= \frac{\Gamma(\beta)\Gamma(\gamma - \beta + 1)}{\Gamma(\gamma)\Gamma(2 - \beta)} a^{1-\beta} M(\gamma - \beta + 1, 2 - \beta, a), \\ \hat{m}_1 &= \frac{\Gamma(\beta)\Gamma(\gamma - \beta + 1)}{\Gamma(\gamma)\Gamma(1 - \beta)} a^{1-\beta} M(\gamma - \beta + 1, 1 - \beta, a). \end{aligned}$$

Proof. Once again, the proof of Theorem 3.1 can be adapted to this case. Now the ladder operators are given by

$$\begin{aligned} \tilde{A}_n(x) &= \tilde{a}_n \left[\sum_{\ell=0}^{\infty} r_n(\ell)r_n(\ell-1) \frac{u(x+1) - u(\ell)}{x+1-\ell} w(\ell) \right. \\ &\quad \left. + t \sum_{\ell=0}^{\infty} r_n(\ell+1-\beta)r_n(\ell-\beta) \frac{u(x+1) - u(\ell+1-\beta)}{x-\ell+\beta} w(\ell+1-\beta) \right], \\ \tilde{B}_n(x) &= \tilde{a}_n \left[\sum_{\ell=0}^{\infty} r_n(\ell)r_{n-1}(\ell-1) \frac{u(x+1) - u(\ell)}{x+1-\ell} w(\ell) \right. \\ &\quad \left. + t \sum_{\ell=0}^{\infty} r_n(\ell+1-\beta)r_{n-1}(\ell-\beta) \frac{u(x+1) - u(\ell+1-\beta)}{x-\ell+\beta} w(\ell+1-\beta) \right]. \end{aligned}$$

They satisfy

$$\tilde{A}_n(x)r_{n-1}(x) - \tilde{B}_n(x)r_n(x) = r_n(x+1) - r_n(x),$$

leading to exactly the same compatibility relations for \tilde{A}_n and \tilde{B}_n as in (3.6)–(3.7). Hence we obtain the same difference relation for the recurrence coefficients \tilde{a}_n and \tilde{b}_n as in the case of the lattice \mathbb{N} or the shifted lattice $\mathbb{N} + 1 - \beta$. The only difference is the initial condition for \tilde{b}_0 , which can again be obtained by calculating \tilde{m}_1/\tilde{m}_0 , where $\tilde{m}_k = m_k + t\hat{m}_k$, with m_k and \hat{m}_k the k -th moment of the measures μ_1 and μ_2 respectively. \square

Notice that letting t tend to 0 or ∞ , we obtain the initial condition for b_0 of Theorem 3.1 and \hat{b}_0 of Theorem 3.2 respectively, which was to be expected. If we use the formula [7, Eq. 13.2.42 on p. 325]

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1, 2-b, z)$$

then the initial condition can also be written as

$$\tilde{b}_0 = \frac{\gamma a}{\beta} \frac{M(\gamma+1, \beta+1, a) - s\Gamma(\gamma-\beta+1)/\Gamma(-\beta)U(\gamma+1, \beta+1, a)}{M(\gamma, \beta, a) - s\Gamma(\gamma-\beta+1)/\Gamma(1-\beta)U(\gamma, \beta, a)}, \quad s = \frac{t}{1+t},$$

where $U(a, b, z)$ is the second solution of the confluent hypergeometric differential equation.

Now we have identified a class of solutions to (3.3)–(3.4), with initial conditions $\tilde{a}_0 = 0$ and (3.28), depending on a parameter t . This initial condition is a homographic function of t . Using the Wronskian formula [7, Eq. 13.2.33] and the contiguous relations [7, Eq. 13.3.15] and

$$\frac{az}{b}M(a+1, b+1, z) = (1-b)(M(a, b, z) - M(a, b-1, z))$$

from [7, Eqs. 13.3.3 and 13.3.4], one can prove that \tilde{b}_0 is an increasing function of $t \in \mathbb{R}^+$ when $0 < \beta < 1$, and a decreasing function of $t \in \mathbb{R}^+$ when $1 < \beta < 2$. Hence for each initial value in $[\tilde{b}_0(0), \tilde{b}_0(\infty)]$ or $[\tilde{b}_0(\infty), \tilde{b}_0(0)]$, the solution to this non-linear system corresponds to recurrence coefficients of orthogonal polynomials on the bi-lattice $\mathbb{N} \cup (\mathbb{N} + 1 - \beta)$.

Exactly the same results concerning the asymptotic behavior of the sum $\sum_{k=0}^{n-1} b_k$ hold as in the case of the generalized Charlier polynomials: the sum behaves as $n^2/4 + \mathcal{O}(n)$ as $n \rightarrow \infty$ when $\beta \neq 1$, and as $n^2/2 + \mathcal{O}(n)$ when $\beta = 1$ or for the limiting cases $t = 0$ and $t = \infty$, which correspond to the lattice \mathbb{N} and $\mathbb{N} + 1 - \beta$. Unfortunately, there seem to be no parameter choices for which the orthogonal polynomials on the bi-lattice turn out to be in a well-known family and for which the recurrence coefficients a_n, b_n are explicitly known (as was the case for $\beta = 1/2, c_1 = c_2$ in the generalized Charlier case). Nevertheless, some parameter choices deserve special attention. The case $\beta = \gamma$ gives rise to the weight function $w(x) = a^x/\Gamma(x+1)$. With the choice $t = 0$ this gives us the Charlier polynomials on the lattice \mathbb{N} . The case $\beta = 1/2, t = 1$ gives the equally spaced lattice $\frac{1}{2}\mathbb{N}$. The weight in the lattice point $k/2$ is then given by

$$w(k/2) = \frac{\Gamma(\gamma + k/2)}{\Gamma(\gamma)} \frac{(2\sqrt{a})^k}{k!},$$

as can be seen by using Legendre's duplication formula [7, Eq. 5.5.5].

4 Concluding remarks

Even though the recurrence coefficients of the generalized Charlier polynomials satisfy the same non-linear system of recurrence equations, their behavior for the bi-lattice is quite different from the behavior on the lattices. In Figure 1 we have plotted on the left the recurrence coefficients a_n and on the right the b_n for three cases. For the a_n (left plot) the lowest curve corresponds to the lattice $\mathbb{N} + 1 - \beta$, the curve just above to the lattice \mathbb{N} and the top curve corresponds to the bi-lattice with $t = 10$ (the parameters are $a = 3$, $\beta = 1/3$). For the b_n (right plot) the top curve corresponds to the lattice $\mathbb{N} + 1 - \beta$, the curve just below to the lattice \mathbb{N} and the lowest curve to the bi-lattice with $t = 10$.

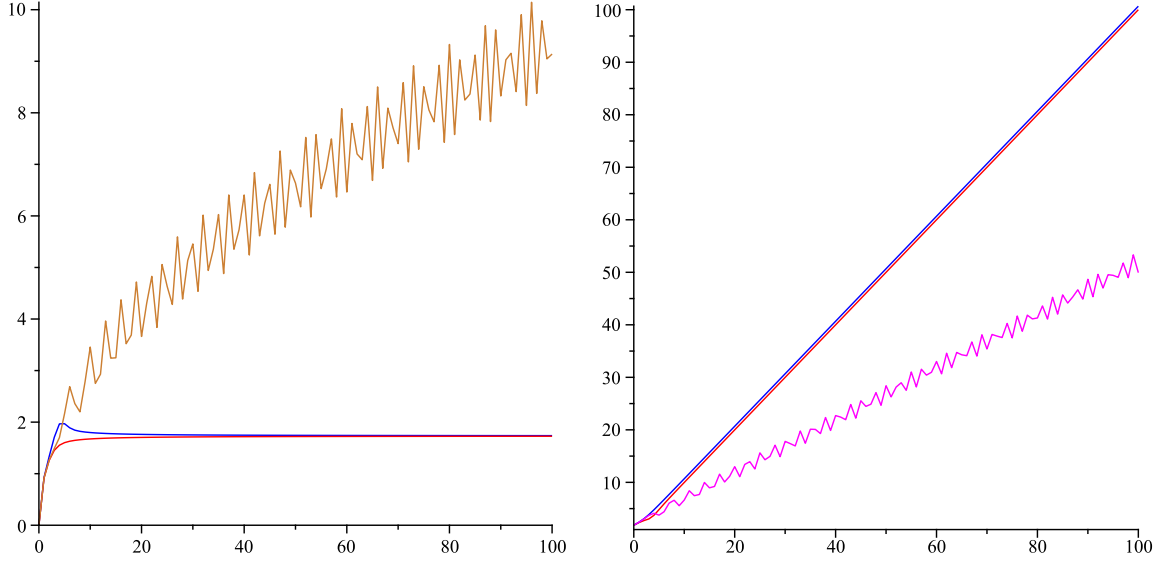


Figure 1: Recurrence coefficients for generalized Charlier polynomials ($a = 3$, $\beta = 1/3$, $t = 10$)

Observe that the behavior for each of the two lattices is quite similar and monotonic as $n \rightarrow \infty$, but for the bi-lattice the behavior is oscillating with a different asymptotic behavior as $n \rightarrow \infty$. We conjecture that for the lattice \mathbb{N} and $\mathbb{N} + 1 - \beta$ one has the asymptotic behavior $\lim_{n \rightarrow \infty} a_n^2 = a$, but for the bi-lattice one has $a_n^2 = n\sqrt{a}/2 + \mathcal{O}(1)$, where the term $\mathcal{O}(1)$ is bounded but oscillating. This behavior is in agreement with the special solution (2.26) for $\beta = 1/2$. For the b_n we conjecture that

$$\lim_{n \rightarrow \infty} b_n - n = \begin{cases} 0 & \text{for the lattice } \mathbb{N}, \\ 1 - \beta & \text{for the lattice } \mathbb{N} + 1 - \beta, \end{cases}$$

and $b_n = n/2 + \mathcal{O}(1)$ for the bi-lattice.

Similar observations hold for the recurrence coefficients of the generalized Meixner polynomials. In Figure 2 we have plotted the recurrence coefficients a_n (on the left) and b_n (on the right) for the same three cases for the parameter values $a = 3$, $\beta = 2/3$, $\gamma = 9/10$. For the a_n , the lowest curve corresponds to the lattice $\mathbb{N} + 1 - \beta$, the curve just above to the lattice \mathbb{N} (both curves are almost identical), and the top curve corresponds to the bi-lattice with $t = 2$. For the b_n the top curve corresponds to the lattice $\mathbb{N} + 1 - \beta$,

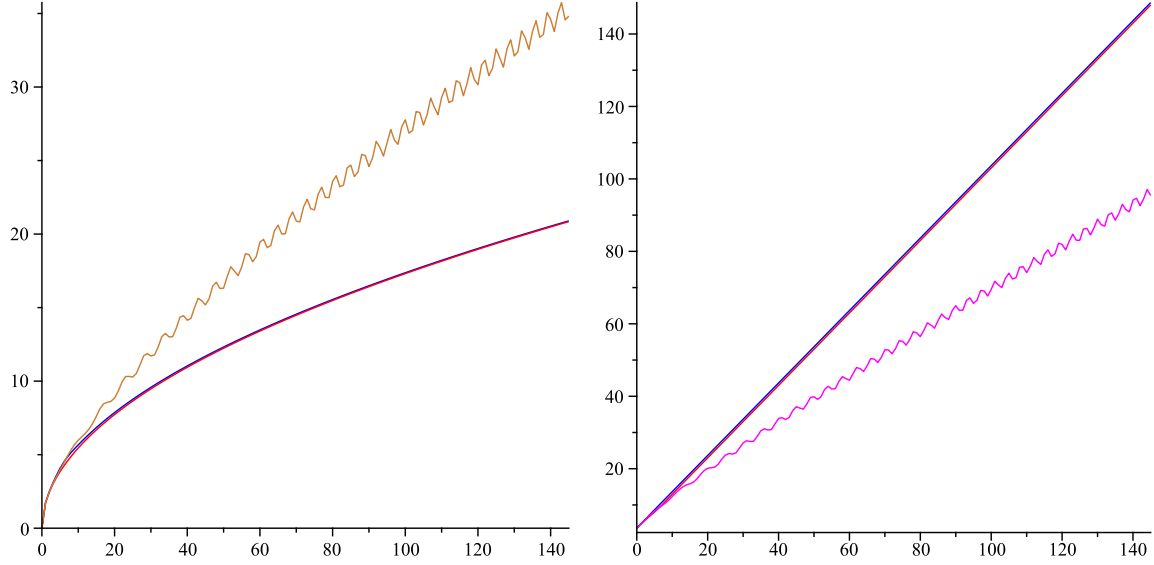


Figure 2: Recurrence coefficients for generalized Meixner polynomials ($a = 3$, $\beta = 2/3$, $\gamma = 9/10$, $t = 2$)

the one just below to the lattice \mathbb{N} (both curves are almost identical), and the lowest curve corresponds to the bi-lattice. Here we conjecture that for each lattice one has

$$\lim_{n \rightarrow \infty} a_n^2 - an = \begin{cases} (\gamma - \beta)a & \text{for the lattice } \mathbb{N}, \\ (\gamma - 1)a & \text{for the lattice } \mathbb{N} + 1 - \beta, \end{cases}$$

and

$$\lim_{n \rightarrow \infty} b_n - n = \begin{cases} a & \text{for the lattice } \mathbb{N}, \\ a + 1 - \beta & \text{for the lattice } \mathbb{N} + 1 - \beta. \end{cases}$$

The asymptotic behavior for the bi-lattice is more difficult and we conjecture $a_n^2/n^{3/2} = \mathcal{O}(1)$ and $b_n/n = \mathcal{O}(1)$, where the $\mathcal{O}(1)$ terms are oscillatory.

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